Compressive Sensing - Exercices - 2022

1 Basic properties of the RIP

1°. Prove that a (ϵ, k)-RIP matrix M satisfies

$$\forall x \in \Sigma_k/\{0\},\ \left|\frac{||Mx||_2^2 - ||x||_2^2}{||x||_2^2}\right| \leq \epsilon \tag{1}$$

2°. Prove that a (ϵ, k)-RIP matrix M satisfies

$$\forall S \subset [\![1, n]\!] \text{ s.t. } |S| \leq k,\ ||M_S' M_S - I_S|| \leq \epsilon \tag{2}$$

2 Convergence of IHT under $(\epsilon, 3s)$-RIP hypothesis

IHT algorithm (Iterative Hard Thresholding) evaluate iteratively a solution of P_0. At each iteration t, the algorithm thresholds each coordinate of k-sparse vector x^t by setting to 0 the lowest $n-k$ coordinates, and keeps unchanged the k highest coordinates (in absolute value).

We define the threshold function H_k as the function from $\mathbb{R}^n$ onto $\mathbb{R}^n$ changing x into a vector x_S where the $n-k$ lowest coordinates are changed to 0.

$$\text{IHT} : \left\{ \begin{array}{l} x(0) = 0 \\ x^t = H_k\left(x^t + M'(y - Mx^t)\right) \end{array} \right. \tag{3}$$

The algorithm exit is

$$\hat{x} = \lim_{t \to +\infty} x^t \tag{4}$$

1°. Let

$$u^t = x^t + M'(y - Mx^t) \tag{5}$$

Prove that

$$y - Mx^t = M(x - x^t) \tag{6}$$
$$u^t = x^t + M'M(x - x^t) \tag{7}$$
$$x^{t+1} = H_k(u^t) \tag{8}$$

2°. Explain why

$$x^{t+1} = \arg\min_{x \in \Sigma_k} ||u^t - x|| \tag{9}$$

the deduce that for all $x \in \mathbb{R}^n$,

$$||u^t - x^{t+1}||^2 \leq ||u^t - x||^2 \tag{10}$$
$$||u^t - x^{t+1}||^2 \leq ||u^t - x^t||^2 \tag{11}$$

3°. Prove that

$$||u^t - x^{t+1}||^2 = ||u^t - x||^2 + ||x^{t+1} - x||^2 ... \tag{12}$$
$$... - 2\left\langle u^t - x, x^{t+1} - x\right\rangle \tag{13}$$
$$\tag{14}$$

4°. Deduce that

$$||x^{t+1} - x||^2 = 2\left\langle (I - M'M)(x^t - x), x^{t+1} - x\right\rangle \tag{15}$$
$$= 2\ \langle \bullet \rangle \tag{16}$$

5°. Nous want now to show that

$$\langle \bullet \rangle \leq \epsilon.||x^t - x||.||x^{t+1} - x|| \tag{17}$$

Let

$$u = x^t - x \tag{18}$$
$$v = x^{t+1} - x \tag{19}$$
$$T = \text{supp}(u) \cup \text{supp}(v) \tag{20}$$

Show that $\text{card}(T) \leq 3k$. Deduce that

$$\langle (I - M'M)u, v\rangle \leq ||(I - M_T' M_T)||_2 ||u_T||_2 ||v_T||_2 \tag{21}$$

and that

$$\langle (I - M'M)u, v\rangle \leq \epsilon ||u_T||_2 ||v_T||_2 \tag{22}$$

6°. Eventually, deduce that

$$||x^{t+1} - x||_2 \leq (2\epsilon)^t ||x||_2 \tag{23}$$

7°. Conclude if $\epsilon < 1/2$.

3 Gaussian concentration inequality around the mean

We want to prove the following result: let $M \in \mathbb{R}^{m \times n}$ a matrix whose coefficients are i.i.d. Gaussian random variables $\mathcal{N}(0, 1/m)$. Then, $\forall x \in \mathbb{R}^n,\ \forall \epsilon \in]0, 1[$,

$$\mathbb{P}\left[\left|||Mx||_2^2 - ||x||_2^2\right| > \epsilon ||x||_2^2\right] \leq 2e^{-Cm\left(\epsilon^2/4 - \epsilon^3/6\right)} \tag{24}$$

Let $\phi_X(t) = \mathbb{E}[e^{tX}]$ the moment generating function of a random variable X.

1°. Recall the expression of $\phi_X(t)$ for a centered Gaussian random variable and for a chi square distribution.

2°. Prove that

$$\mathbb{E}\left[||Mx||_2^2\right] = ||x||_2^2 \tag{25}$$

Let $y = Mx$, $\gamma = ||x||_2^2/m$, and $z_i = y_i^2 - \gamma$.

3°. Prove that

$$||Mx||_2^2 - ||x||_2^2 = \sum_{i=1}^{m} z_i = S_m \tag{26}$$

4°. Prove that

$$\mathbb{P}\left[\left|||Mx||_2^2 - ||x||_2^2\right| > \epsilon ||x||_2^2\right] = ... \tag{27}$$

$$... = \mathbb{P}\left[S_m > \epsilon m \gamma\right] + \mathbb{P}\left[S_m < -\epsilon m \gamma\right] \tag{28}$$

5°. Using Markov inequality, prove that

$$\mathbb{P}\left[S_m > \epsilon m \gamma\right] \leq \frac{\mathbb{E}\left[e^{tS_m}\right]}{e^{t\epsilon m \gamma}} \tag{29}$$

Deduce that

$$\mathbb{P}\left[S_m > \epsilon m \gamma\right] \leq \phi_{z_i}(t)^m e^{-mt\epsilon\gamma} = p(t)^m \tag{30}$$

with $p(t) = \phi_{z_1}(t) \exp(-t\epsilon\gamma)$.

6°. Prove that

$$p(t) = \frac{e^{-t(1+\epsilon)\gamma}}{\sqrt{1 - 2\gamma t}} \tag{31}$$

7°. Prove that $p(t)$ is minimum when

$$t = \frac{1}{2\gamma}\frac{\epsilon}{1+\epsilon} \tag{32}$$

Then deduce that

$$\mathbb{P}\left[S_m > \epsilon m \gamma\right] \leq \exp\left(-m\left(\epsilon^2/4 - \epsilon^3/6\right)\right) \tag{33}$$

and

$$\mathbb{P}\left[S_m < -\epsilon m \gamma\right] \leq \exp\left(-m\left(\epsilon^2/4 - \epsilon^3/6\right)\right) \tag{34}$$

8°. Conclude and make the link with RIP property.