$n \geq 3$; $\alpha, \beta > 0$ $\quad f(t) = \frac{\alpha\beta^\alpha}{t^{\alpha+1}} \mathbb{1}_{[t \geq \beta]}$

$X \sim \mathcal{P}(\alpha,\beta)$

1°. $F(t) = 1 - (\beta/t)^\alpha \quad t \geq \beta$

$\mathbb{E}[X] = \frac{\alpha\beta}{\alpha-1} \quad \alpha>1 \qquad \mathbb{V}(X) = \frac{\alpha\beta^2}{(\alpha-1)^2(\alpha-2)} \quad \alpha>2$

2°. En ut. les fonc. de répartition, $Y = \ln(X/\beta) \sim \mathcal{E}(\alpha)$

$F_Y(x) = \mathbb{P}[Y \leq x] = \mathbb{P}[X \leq \beta e^x] = 1 - e^{-\alpha x}$

3°. $\alpha>2$; $Z_n = \min(X_i)$

$F_{Z_n}(t) = 1 - \mathbb{P}[\min X_i \geq t] = 1 - \mathbb{P}\left(\bigcap_{i=1}^n [X_i \geq t]\right)$

$= 1 - \mathbb{P}[X_1 \geq t]^n = 1-(1-\mathbb{P}[X_1 \leq t])^n$

$= 1 - (1-1+(\beta/t)^\alpha)^n = 1-(\beta/t)^{n\alpha} \sim \mathcal{P}(n\alpha,\beta)$

$\mathbb{E}[Z_n] = \frac{n\alpha\beta}{n\alpha-1} \qquad \mathbb{V}(Z_n) = \frac{n\alpha\beta^2}{(n\alpha-1)^2(n\alpha-2)} \qquad \hat Z_n = \left(1-\frac{1}{n\alpha}\right)Z_n$

$\mathbb{E}[c_n Z_n] = \beta \iff c_n = \frac{n\alpha-1}{n\alpha} = 1-\frac{1}{n\alpha}$

4°. $\mathbb{E}[Z_n] \to \beta$ et $Var(Z_n) \to 0$. $\qquad \mathbb{V}(\hat Z_n) = \frac{\beta^2}{\alpha}\frac{1}{n(\alpha n-2)} \underset{\infty}{\sim} \frac{\beta^2}{n^2\alpha^2}$

$\Rightarrow \hat Z_n \xrightarrow[L^2]{n} \beta$

5°. $W_n = n\left(\sum \ln(X_k/\beta)\right)^{-1}$

Calculer $\mathbb{E}[W_n]$ et mq $\hat W_n = \frac{n-1}{n} W_n$ est sans biais de α

par ⊥⊥ des X_i, $\sum \ln(X_k/\beta) \sim \Gamma(n,\alpha)$ soit $V_n = \sum \ln(X/\beta)$

$\frac1n \mathbb{E}[W_n] = \mathbb{E}[V_n^{-1}] = \int_0^\infty \frac1x \frac{1}{\Gamma(n)} e^{-\alpha x} x^{n-1}\alpha^n \, dx = \frac{\alpha}{n-1}$ $\qquad$ • $\hat W_n = \frac{n-1}{n}W_n$

$\mathbb{E}[W_n] = \frac{n}{n-1}\alpha \Rightarrow \mathbb{E}[\hat W_n] = \frac{n-1}{n}\times\frac{n}{n-1}\alpha = \alpha$

$\mathbb{E}[\hat W_n^2] = \left(\frac{n-1}{n}\right)^2 \frac{n^2\alpha^2}{(n-1)(n-2)} = \frac{n-1}{n-2}\alpha^2$

6°. $Var(\hat W_n)$

$\frac{1}{n^2}\mathbb{E}[W_n^2] = \mathbb{E}[V_n^{-2}] = \frac{\alpha^2}{(n-1)(n-2)} \Rightarrow$

$Var(\hat W_n) = \frac{\alpha^2}{n-2} \xrightarrow[n\to\infty]{} 0$ $\quad$ car $\mathbb{V}(W_n) = \frac{n^2\alpha^2}{(n-1)(n-2)} - \left(\frac{n\alpha}{n-1}\right)^2$

$= \frac{n^2\alpha^2}{(n-1)^2(n-2)}$

7°. IC_α. $\alpha = 5\%$

Nous reprendrons à cette question dans le chapitre 3...